

CATEGORIES OF LAYERED SEMIRINGS

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ABSTRACT. We generalize the constructions of [17, 19] to layered semirings, in order to enrich the structure and provide finite examples for applications in arithmetic (including finite examples). The layered category theory of [19] is extended accordingly, to cover noncancellative monoids.

1. INTRODUCTION

This paper is a continuation of [17] and [19]. Tropical mathematics often involves the study of valuations, whose targets are ordered Abelian groups that can be viewed as max-plus algebras. The layered supertropical domain was introduced in [17], and put in a categorical framework in [19], in order to provide algebraic tools with which to study this structure.

Tropical mathematics often involves the study of valuations, whose targets are ordered Abelian groups that can be viewed as max-plus algebras. The basic functor used in [19] goes from the category of cancellative ordered Abelian monoids to the category of L -layered domains[†] with respect to a semiring[†] L . (We use the generic notation [†] to indicate that we do not require a zero element.)

On the other hand, many important classical arithmetical results are proved by passing to finite structures (i.e., modulo a prime number). The main objective of this paper is to open the way to an arithmetic tropical theory, by permitting finite tropical structures. This might seem to be an oxymoron, since all nontrivial ordered groups are infinite. But valuation theory has been enriched in [11] and [31] by means of valuations to arbitrary ordered Abelian monoids, thereby raising the possibility of a layered semiring[†] construction for any ordered Abelian monoid.

Since any cancellative ordered monoid is necessarily infinite, we need to include noncancellative monoids in our category if we want to deal with finite structures and their corresponding arithmetic. But then, as observed already in [17], the naive analog of [17, Construction 3.2] does not satisfy distributivity, so we must turn to a more sophisticated version, given below in Construction 3.5. This requires a 0-layer, i.e., $0 \in L$, at the cost of a decidedly more complicated multiplication. So at the outset we consider ‘absorption’ via the elements 0 and ∞ .

The dividend is far greater flexibility in our examples, cast in a more general categorical setting than given in [19]. Construction 3.5 is verified in Theorem 3.6. In the process we obtain finite structures, as indicated in Example 4.19. Namely, applying “truncation” both to the given valued monoid and the sorting set yields finite examples and could permit one to apply the corresponding arithmetic tools. Since the 0 layer and infinite layer both play significant roles in this theory, we study their properties and interactions in detail in §4.

Several serious technical difficulties arise when we try to put this more general construction in its categorical context, because a homomorphism of monoids might send a noncancellative monoid to a cancellative monoid, thereby requiring us to switch back and forth from one construction to the other.

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The corresponding maps apparently cannot be written as morphisms of semirings[†], so one must broaden either the class of morphisms or the class of objects in the category.

We try both approaches in turn, the first approach occupying the body of this paper and the second approach discussed in the appendix. In §4 we introduce our main examples. In §5, which still is not cast in full generality, we pass from the category of valued monoids to the category of semirings[†] by means of the “0-excepted” homomorphisms of Definition 5.14. This permits us in §6 to describe the tropicalization functor more generally, for rings that need not be integral domains. Furthermore, if one turns to the basic link of tropical geometry with classical algebraic geometry via valuations, one is led to consider more general “transmissions” which pass from valuation to valuation.

The key to tying this in with tropicalization is Kapranov’s Lemma. Elaborating on [19, §8], we show in Remark 6.6 how Kapranov’s Lemma can be expressed in terms of a **Kapranov map**, thereby yielding a “layering” functor for polynomial functions. This map is compatible with the tropicalization map given in [27].

In §7 we see that the category LaySemi^\dagger of layered semirings ties in to layered supervaluations, and specializes to the category STROP from [20], when we take $L = \{0, 1, \infty\}$ and $R_0 = \{0_R\}$. The key result in this regard is Theorem 7.8 and its corollary, which show that the transmissions of layered supervaluations often become layered homomorphisms under certain natural assumptions.

In Appendix A (§8) even fuller generality is obtained by considering structures more general than semirings[†], analogous to the supertropical monoids of [20]. Here the noncancellative products belong to the 0-layer, for which addition with the rest of the structure is not defined; as a result, we do not quite have a semiring[†].

2. BACKGROUND

For us, a monoid is a multiplicative semigroup with a unit element $1_{\mathcal{M}}$. We work with semirings and their (multiplicative) monoids.

2.1. Semigroups and semirings. We review a few definitions from semigroups and semirings. We say that an element a of a semigroup $\mathcal{M} := (\mathcal{M}, \cdot)$ is **partially absorbing** if $ab = a$ for some $b \in \mathcal{M}$; an element a of $\mathcal{M} := (\mathcal{M}, \cdot)$ is **absorbing** if $ab = ba = a$ for all $b \in \mathcal{M}$.

Lemma 2.1. *If \mathcal{M} is an Abelian semigroup with a unique partially absorbing element a , then a is absorbing.*

Proof. Suppose $ab = a$. For any $c \in \mathcal{M}$ we have $(ca)b = c(ab) = ca = ac$. Thus, ac is partially absorbing, implying $ac = a$ by hypothesis. \square

Usually, the absorbing element is identified with 0 , but it could also be identified with the **infinite element** ∞ , given by

$$\infty \cdot a = a \cdot \infty = \infty, \quad \text{for all } 0 \neq a \in \mathcal{M}. \quad (2.1)$$

(We do not necessarily assume that \mathcal{M} contains 0 or ∞ . The partially absorbing element ∞ is absorbing when $0 \notin \mathcal{M}$.)

A semigroup \mathcal{M} is **pointed** if it has an absorbing element $0_{\mathcal{M}}$.

A semigroup \mathcal{M} is **cancellative** with respect to a subset S if $as = bs$ implies $a = b$ whenever $a, b \in \mathcal{M}$ and $s \in S$. A pointed semigroup \mathcal{M} is **cancellative** if \mathcal{M} is cancellative with respect to $\mathcal{M} \setminus \{0_{\mathcal{M}}\}$.

An element ∞ in a semiring[†] R is **infinite** if it is absorbing with respect to addition, i.e., satisfies

$$r + a = r \quad \text{for all } a \in R. \quad (2.2)$$

Definition 2.2. A **domain**[†] is a semiring[†] R that is cancellative under multiplication. A semiring[†] R is a **domain** if $R \cup \{0\}$ is a domain[†]. Likewise, R is a **semifield**[†] if R is closed under multiplication. R is a **semifield** if $R \cup \{0\}$ is a semifield[†].

Although we have two usages for ‘infinite,’ one additive and one multiplicative, they are connected by the following observation:

Proposition 2.3. *If $R_\infty = R \cup \{\infty\}$ where R is a semifield[†] and $\infty \in R_\infty$ is an infinite element in the sense of (2.2), then $a := \infty$ also satisfies (2.1).*

Proof. $a = a + ab = (ab^{-1} + a)b = ab$. \square

Congruences over semifields[†] are described in detail in [13]. (The domains[†] of eventual tropical interest to us are polynomial semirings[†] over semifields[†], which are needed to define tropical varieties, as described in [17, 19].)

As in [19] we work with the category Semir^\dagger of semirings[†] and their homomorphisms, as compared to the category Semir of semirings and semiring homomorphisms. We refer the reader to [19] for preliminary facts that we need; an earlier reference is [5]. As noted in [19], the category Semir^\dagger is isomorphic to a subcategory of the category Semir , since any semiring[†] R can be embedded in a semiring $R \cup \{0\}$ by formally adjoining a zero element 0.

2.1.1. Pre-ordered semigroups and semirings[†].

Definition 2.4. A semigroup $\mathcal{M} := (\mathcal{M}, \cdot)$ (or a monoid $\mathcal{M} := (\mathcal{M}, \cdot, \mathbb{1}_{\mathcal{M}})$) is **pre-ordered** (resp. **partially pre-ordered**, **partially ordered**, **ordered**) if it has a pre-order \leq (resp. partially pre-order, partial order, order) such that

$$b \leq c \quad \text{implies} \quad ab \leq ac \quad \text{and} \quad ba \leq ca, \quad \forall a \in \mathcal{M}. \quad (2.3)$$

As in [19], we assume that all preorders are positive. PPreOMon , PreOMon , POMon , OMon , and OMon^+ denote the respective categories of partially pre-ordered, pre-ordered, partially ordered, ordered, and cancellative ordered monoids, whose morphisms are the order-preserving homomorphisms.

The crucial observation here is that any semiring[†] becomes a partially pre-ordered semigroup via the rule (also cf. [13]):

$$a \leq b \quad \text{iff} \quad a = b \quad \text{or} \quad b = a + c \quad \text{for some } c \in R. \quad (2.4)$$

We say that a semiring[†] R is **pre-ordered** (resp. **partially pre-ordered**, **partially ordered**, **ordered**) if it has a partial pre-order \leq (resp. partial order, order) with respect to which both the monoid $(R, \cdot, \mathbb{1}_R)$ and the semigroup $(R, +)$ satisfy Condition (2.3) of Definition 2.4.

By [19, Proposition 3.9], there is a natural functor $\text{Semir}^\dagger \rightarrow \text{PPreOMon}$, where we define the partial pre-order on a semiring[†] R as in (2.4).

2.2. Valued monoids. Although we focused on ordered monoids in [19], tropical mathematics is concerned with valuations. More generally, we can take the target to be a monoid, cf. [16, Definition 2.1].

Definition 2.5. A monoid $\mathcal{M} := (\mathcal{M}, \cdot, \mathbb{1}_{\mathcal{M}})$ is **m-valued** with respect to an ordered monoid $\mathcal{G} := (\mathcal{G}, \cdot, \geq, \mathbb{1}_{\mathcal{G}})$ if there is an onto monoid homomorphism $v : \mathcal{M} \rightarrow \mathcal{G}$. (In other words, $v(ab) = v(a)v(b)$.) We also call v an **m-valuation**. We notate this set-up as the **triple** $(\mathcal{M}, \mathcal{G}, v)$.

This fits in better with our algebraic notation for semirings[†]. Thus, any valuation $v : K \rightarrow \mathcal{G}$ is an m-valuation, where we just disregard addition in K . The hypothesis that v is onto can always be attained by replacing \mathcal{G} by $v(\mathcal{M})$ if necessary.

The category of triples should be quite intricate, since the morphisms should include all maps which “transmit” one m-valuation to another. We explore this idea further in §7, but for the most part take a simpler approach, following [19].

Definition 2.6. ValMon is the category of valued monoids whose objects are triples $(\mathcal{M}, \mathcal{G}, v)$ as in Definition 2.5, for which a morphism

$$\phi : (\mathcal{M}, \mathcal{G}, v) \longrightarrow (\mathcal{M}', \mathcal{G}', v') \quad (2.5)$$

is comprised of a pair $(\phi_{\mathcal{M}}, \phi_{\mathcal{G}})$ of a monoid homomorphism $\phi_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M}'$, as well as an order-preserving monoid homomorphism $\phi_{\mathcal{G}} : \mathcal{G} \rightarrow \mathcal{G}'$, satisfying the compatibility condition

$$v'(\phi_{\mathcal{M}}(a)) = \phi_{\mathcal{G}}(v(a)), \quad \forall a \in \mathcal{M}. \quad (2.6)$$

Remark 2.7. When the value map v of the triple $(\mathcal{M}, \mathcal{G}, v)$ is 1:1, then \mathcal{M} inherits the order from \mathcal{G} , by stipulating that $a < b$ when $v(a) < v(b)$. In this way, we can view OMon as a full subcategory of ValMon .

2.3. Congruences. Since we work in the framework of universal algebras, we need some general observations, and then specialize to the cases of interest to us (semigroups and semirings). One defines a **congruence** Ω of an algebraic structure \mathcal{A} to be an equivalence relation \equiv which preserves all the relevant operations and relations; we call \equiv the **underlying equivalence** of Ω . Equivalently, a congruence Ω is a sub-structure of $\mathcal{A} \times \mathcal{A}$ that contains the diagonal $\text{diag}(R) := \{(a, a) : a \in R\}$, as described in Jacobson [23, §2].

Since the most important semirings[†] for us are domains[†], we want to know, given a congruence Ω on R , when the factor semiring[†] R/Ω has an absorbing element, and when it is a domain[†]. Given a subset $A \subset R$, we write $b \equiv A$ if $b \equiv a$ for some $a \in A$. We call an ideal $\mathfrak{a} \triangleleft R$ **closed** under Ω if $b \equiv \mathfrak{a}$ implies $b \in \mathfrak{a}$.

Lemma 2.8. *Suppose Ω is a congruence on a semiring[†] R .*

- (i) *R/Ω is a domain[†] iff its underlying equivalence \equiv is **cancellative**, in the sense that $ab \equiv ac$ implies $b \equiv c$.*
- (ii) *If R/Ω is a semiring with absorbing element, which we denote as $\bar{0}$, then the pre-image I of $\bar{0}$ is a closed ideal of R all of whose elements are equivalent. Conversely, if \mathfrak{a} is a closed ideal of R all of whose elements are equivalent, then the image of \mathfrak{a} is the absorbing element of R/Ω .*
- (iii) *When (ii) holds, R/Ω is a domain iff \equiv is cancellative with respect to all elements not in \mathfrak{a} , in the sense that if $ab \equiv ac$ for $a \notin \mathfrak{a}$, then $b \equiv c$.*

Proof. Write \bar{a} for the image of a in R/Ω .

- (i) $ab \equiv ac$ iff $\bar{a}\bar{b} = \bar{a}\bar{c}$, iff $\bar{b} = \bar{c}$, iff $b \equiv c$.
- (ii) If $a, b \in I$, then $\bar{a} = \bar{b} = \bar{0}$, implying $a \equiv b$. Conversely, if \mathfrak{a} is a closed ideal of R all of whose elements are equivalent, then the image of \mathfrak{a} is an ideal of R/Ω consisting of a single element, which must thus be the absorbing element.
- (iii) The condition translates to saying that $\bar{a}\bar{b} = \bar{a}\bar{c}$ for $\bar{a} \neq \bar{0}$ implies $\bar{b} = \bar{c}$.

□

It is useful to weaken the notion of congruence.

Definition 2.9. A **half-congruence** Ω is a sub-structure of $\mathcal{A} \times \mathcal{A}$ that contains the diagonal and is transitive in the sense that if Ω contains (a, b) and (b, c) then it also contains (a, c) .

Throughout the body of this paper R denotes a commutative semiring[†].

Example 2.10. In the language of monoids, if $\mathfrak{a}_1, \mathfrak{a}_2$ are monoid ideals of a monoid $\mathcal{M} := (\mathcal{M}, \cdot)$, then

$$(\mathfrak{a}_1 \times \mathfrak{a}_2) \cup \{(a, a) : a \in \mathcal{M}\}$$

is a congruence since $\mathfrak{a}_i a \subseteq \mathfrak{a}_i$. But in the language of semirings[†], if $\mathfrak{a}_1, \mathfrak{a}_2$ are semiring[†] ideals of a semiring[†] R , then $(\mathfrak{a}_1 \times \mathfrak{a}_2) \cup \{(r, r) : r \in R\}$ need not even be a half-congruence, since it may not be closed under addition. (In general, $\mathfrak{a}_i + r \not\subseteq \mathfrak{a}_i$.)

Lemma 2.11. A transitive relation \sim is a half-congruence on a semiring[†] if it is closed under addition and multiplication by the diagonal, i.e., if it satisfies the following conditions for all a_1, a_2 , and b :

$$\begin{aligned} a_1 \sim a_2 & \quad \text{implies} \quad a_1 + b \sim a_2 + b; \\ a_1 \sim a_2 & \quad \text{implies} \quad a_1 b \sim a_2 b. \end{aligned} \tag{2.7}$$

Proof. $a_1 + b_1 \sim a_2 + b_1 = b_1 + a_2 \sim b_2 + a_2 = a_2 + b_2$. Likewise for multiplication. □

3. THE LAYERED STRUCTURE

We are ready to bring in the leading players in this theory, taking into account a 0-layer.

Definition 3.1. A pre-order is **directed** if for any a, b there is c such that $c \geq a$ and $c \geq b$.

We assume throughout that the sorting set L is a directed, (non-negative) pre-ordered semiring semiring[†] with zero element $0 := 0_L$; the bulk of our applications in this paper are for L ordered. Let $L^\times := L \setminus \{0\}$. We recall [17, Construction 3.2].

Construction 3.2. Suppose \mathcal{G} is a given cancellative monoid. $R := \mathcal{R}(L^\times, \mathcal{G})$ is defined set-theoretically as $L^\times \times \mathcal{G}$, where we denote the element (ℓ, a) as $^{[\ell]}a$ and, for $k, \ell \in L$, $a, b \in \mathcal{G}$, we define multiplication componentwise, i.e.,

$$^{[k]}a \cdot ^{[\ell]}b = ^{[k\ell]}ab. \quad (3.1)$$

Addition is given by the rules:

$$^{[k]}a + ^{[\ell]}b = \begin{cases} ^{[k]}a & \text{if } a > b, \\ ^{[\ell]}b & \text{if } a < b, \\ ^{[k+\ell]}a & \text{if } a = b. \end{cases} \quad (3.2)$$

We define $R_\ell := \{\ell\} \times \mathcal{G}$, for each $\ell \in L^\times$. Namely $R = \bigcup_{\ell \in L} R_\ell$.

This is our prototype of a layered pre-domain[†], and should be borne in mind throughout the sequel. Note that in this case R_1 is a monoid, which is isomorphic to \mathcal{G} .

Nevertheless, we also consider the possibility that the monoid \mathcal{G} is noncancellative, in which case, as noted in [17], Construction 3.2 fails to satisfy distributivity and thus is not a semiring.

Definition 3.3. Suppose \mathcal{G} is an ordered Abelian monoid. An element $z \in \mathcal{G}$ is a **noncancellative product** if $z = ab = ac$ for suitable a, b, c with $b \neq c$.

More generally, when $(\mathcal{M}, \mathcal{G}, v)$ is a triple, an element $z \in \mathcal{M}$ is a **v-noncancellative product** if $v(z) = v(ab) = v(ac)$ for suitable a, b, c , where $v(b) \neq v(c)$.

Proposition 3.4. The set A of v-noncancellative products comprises a monoid ideal of \mathcal{M} .

Proof. If $v(z) = v(ab) = v(ac) \in A$, then $v(ad)v(c) = v(ac)v(d) = v(zd) = v(abd) = v(ad)v(b)$. \square

Construction 3.5. Suppose $(\mathcal{M}, \cdot, \geq, \mathbb{1}_{\mathcal{G}})$ is an Abelian monoid, with an m -valuation $v : \mathcal{M} \rightarrow \mathcal{G}$, and \mathfrak{a} is a monoid ideal of \mathcal{M} containing all v-noncancellative products. $R := \mathcal{R}(L, \mathcal{M})_{\mathfrak{a}}$ is defined set-theoretically as $(L^\times \times (\mathcal{M} \setminus \mathfrak{a})) \cup (\{0\} \times \mathfrak{a})$, where we denote the element (ℓ, a) as $^{[\ell]}a$ and, for $k, \ell \in L$, $a, b \in \mathcal{M}$, multiplication is defined componentwise, i.e., via the rules:

$$^{[k]}a \cdot ^{[\ell]}b = \begin{cases} ^{[k\ell]}ab & \text{if } ab \notin \mathfrak{a}, \\ ^{[0]}ab & \text{if } ab \in \mathfrak{a}. \end{cases} \quad (3.3)$$

Addition is given as in Construction 3.2.

$$^{[k]}a + ^{[\ell]}b = \begin{cases} ^{[k]}a & \text{if } v(a) > v(b), \\ ^{[\ell]}b & \text{if } v(a) < v(b), \\ ^{[k+\ell]}a & \text{if } v(a) = v(b). \end{cases} \quad (3.4)$$

$R_0 := \{0\} \times \mathfrak{a}$ and $R_\ell := \{\ell\} \times (\mathcal{M} \setminus \mathfrak{a})$, for each $\ell \in L^\times$. Thus, $R = \bigcup_{\ell \in L} R_\ell$.

This encompasses the case where $\mathcal{M} = \mathcal{G}$ is an ordered monoid and v is the identity map. We usually refer to this special case, in the interest of clarity.

Theorem 3.6. $R := \mathcal{R}(L, \mathcal{M})_{\mathfrak{a}}$ is a semiring[†], while R is a semiring iff the monoid \mathcal{M} is pointed, in which case $\mathbb{0}_R = ^{[0]}\mathbb{0}_{\mathcal{M}}$.

$R \setminus R_0$ is a semiring[†] iff \mathfrak{a} is prime as a monoid ideal of \mathcal{M} .

Proof. The verification that R is a semiring[†] was essentially done in [17, Proposition 3.3]. The trickiest part again is to verify the distributivity law

$$x(y + z) = xy + xz.$$

Write $x = ^{[k]}a$, $y = ^{[\ell]}b$, and $z = ^{[m]}c$, and assume that $v(b) \geq v(c)$. If $v(ab) > v(ac)$, then clearly $v(b) > v(c)$, and

$$x(y + z) = xy = xy + xz.$$

Thus we are done unless $v(ab) = v(ac)$.

If $v(b) = v(c)$ with $ab \notin \mathfrak{a}$, then

$$x(y + z) = ^{[k]}a (^{[\ell]}b + ^{[m]}b) = ^{[k]}a ^{[\ell+m]}b = ^{[k\ell+k m]}(ab) = ^{[k\ell]}(ab) + ^{[k m]}(ab) = xy + xz.$$

If $v(b) = v(c)$ with $ab \in \mathfrak{a}$, then

$$x(y + z) = [k]_a ([\ell]b + [m]b) = [k]_a [\ell+m]b = [0](ab) = [0](ab) + [0](ab) = xy + xz.$$

If $v(b) > v(c)$, then $ab, ac \in \mathfrak{a}$, so

$$x(y + z) = [k]_a ([\ell]b + [m]c) = [k]_a [\ell]b = [0](ab) = [0](ab) + [0](ac) = xy + xz.$$

When \mathcal{M} is pointed, the verification of the zero element is an easy computation.

The next assertion is clear: $xy \in \mathfrak{a}$ iff $xy \in R_0$. \square

We have the maps $\nu_{\ell,k} : R_k \rightarrow R_\ell$ given by $\nu_{\ell,k}([k]_a) = [\ell]_a$ for any $0 < k \leq \ell$, and a **sorting map** $s : R \rightarrow L$ given by $s(\ell, a) = \ell$, for any $a \in \mathcal{M}$, $\ell \in L$.

Note that $R \setminus \mathfrak{a}$ could be a finite set, in which case we could apply various arithmetic tools such as zeta functions.

Remark 3.7. If $\mathcal{M} = \mathcal{G}$ and $\mathfrak{a} = \emptyset$, then $R_0 = \emptyset$, and $\mathcal{R}(L, \mathcal{G})$ coincides with the semiring[†] $\mathcal{R}(L^\times, \mathcal{G})$ of Construction 3.2.

Lemma 3.8. For any multiplicative idempotent ℓ of L , the subset $R_0 \cup R_\ell$ of $\mathcal{R}(L, \mathcal{G})$ is a monoid, together with a natural homomorphism to \mathcal{G} .

Proof. If $[\ell]_a, [\ell]_b \in R_1$, then their product either is $[\ell^2](ab) = [\ell](ab) \in R_\ell$, or $[0](ab)$ if $ab \in \mathfrak{a}$. The natural homomorphism is given by $[\ell]_a \mapsto v(a)$. \square

The main application of this lemma is for $\ell = 1$. The layer R_1 is of particular importance, since its unit element is 1_R . Two other obvious multiplicative idempotents of L are 0 and ∞ (when appropriate, since ∞ need not belong to L).

4. LAYERED SEMIRINGS[†]

In this section we provide the framework for Construction 3.5 and truncation (Example 4.19). We deal with a zero layer, i.e., assume that $0 \in L$, and treat the zero component R_0 specially, taking the opportunity to fit the zero element of R (if it exists) into the theory. Since we also want to consider monoids that are not cancellative, we need to work harder to obtain distributivity. We axiomatize in order to place the theory in a categorical framework.

Definition 4.1. Suppose (L, \geq) is a partially pre-ordered, directed semiring. An *L -layered semiring[†]*

$$R := (R, L, s, (\nu_{m,\ell})),$$

is a semiring[†] R , together with a family $\{R_\ell : \ell \in L\}$ of disjoint subsets $R_\ell \subset R$, such that

$$R := \bigcup_{\ell \in L} R_\ell, \tag{4.1}$$

and a family of **sort transition maps**

$$\nu_{m,\ell} : R_\ell \rightarrow R_m, \quad \forall m \geq \ell > 0,$$

such that

$$\nu_{\ell,\ell} = \text{id}_{R_\ell}$$

for every $\ell \in L$, and

$$\nu_{m,\ell} \circ \nu_{\ell,k} = \nu_{m,k}, \quad \forall m \geq \ell \geq k,$$

whenever both sides are defined. To avoid complications, we assume that any element of R_0 can be written as a product ab where $a, b \in R \setminus R_0$. We also require the axioms A1–A4, and B, given presently, to be satisfied. (In order to have our definition compatible with the L -layered pre-domains[†] of [17], we permit $R_0 = \emptyset$.)

We also require R_∞ to be the direct limit of the R_ℓ , $\ell > 0$, together with maps $\nu_{\infty,\ell} : R_\ell \rightarrow R_\infty$, which extend to a map $\nu : R \rightarrow R_\infty$. (For $c = ab \in R_0$ we define $\nu(c) = \nu(a)\nu(b)$.)

We write a^ν for $\nu(a)$. We write $a \cong_\nu b$ for $a \in R_k$ and $b \in R_\ell$ whenever $a^\nu = b^\nu$, which means $\nu_{m,k}(a) = \nu_{m,\ell}(b)$ in R_m for some $m \geq k, \ell$. (This notation is used generically: We write $a \cong_\nu b$ even when the sort transmission maps are denoted differently.)

Similarly, we write $a \leq_\nu b$ if $a^\nu + b^\nu = b^\nu$, which means $\nu_{m,k}(a) + \nu_{m,\ell}(b) = \nu_{m,\ell}(b)$ in R_m for some $m \geq k, \ell$, and we write $a <_\nu b$ if $a \leq_\nu b$ but $a \not\equiv_\nu b$.

The axioms are as follows:

- A1. $1_R \in R_1$.
- A2. If $a \in R_k$ and $b \in R_\ell$, then $ab \in R_{k\ell} \cup R_0$.
- A3. The product in R is compatible with sort transition maps: Suppose $a \in R_k$, $b \in R_\ell$, with $m \geq k$ and $m' \geq \ell$. Then
- $$\nu_{m,k}(a) \cdot \nu_{m',\ell}(b) = \nu_{mm',k\ell}(ab).$$
- A4. $\nu_{\ell,k}(a) + \nu_{\ell',k}(a) = \nu_{\ell+\ell',k}(a)$ for all $a \in R_k$ and all $\ell, \ell' \geq k$.
- A5. If $a \in R_k$, $b \in R_\ell$, and $c = a + b \in R_{k'}$, then
- $$\nu_{m,k'}(c) = \nu_{m,k}(a) + \nu_{m,\ell}(b)$$
- for each $m \geq k + \ell$.
- A6. R_0 is an additive semigroup (and thus an ideal) of R .

- B. (Supertropicality) Suppose $a \in R_k$, $b \in R_\ell$, and $a \cong_\nu b$. Then $a + b \in R_{k+\ell}$ with $a + b \cong_\nu a$. If moreover $k = \infty$, then $a + b = a$.

We say that any element a of R_k has **sort** k ($k \in L$). L is called the **sorting semiring** of the layered semiring[†] $R = \bigcup_{\ell \in L} R_\ell$. Thus, ℓ is the **sort** of the **layer** R_ℓ .

The **sorting map** $s : R \rightarrow L$ is the map that sends every element $a \in R_\ell$ to its sort ℓ .

(Taken from [19, Definition 5.2]) An L -layered **pre-domain**[†] is an L -layered semiring[†] in which Axiom A2 is strengthened to the condition $ab \in R_{k\ell}$. An L -layered semiring[†] $R := (R, L, s, (\nu_{m,\ell}))$ is called **uniform** when the sorting semiring[†] L is totally ordered and the sort transition maps $\nu_{\ell,k}$ are bijective for each $\ell > k > 0$.

Definition 4.2. An L -layered **pre-domain**[†] is an L -layered semiring[†] R for which R_1 is a monoid.

Definition 4.3. An L -layered semiring[†] is **ν -bipotent** if $a + b \in \{a, b\}$ whenever $a \not\equiv_\nu b$.

An L -layered **bi-domain**[†] is a ν -bipotent L -layered domain[†].

Remark 4.4. For layered bi-domains[†], Axiom A5 says that $a <_\nu b$ implies $\nu_{m,k}(a) <_\nu \nu_{m,\ell}(b)$.

Let us put Construction 3.5 into context, using the layered version of Definition 3.3. An element $z \in R$ is a **ν -noncancellative product** if $z^\nu = a^\nu b^\nu = a^\nu c^\nu$ for suitable a, b, c , where $b \not\equiv_\nu c$. Note that the set of ν -noncancellative products of an L -layered semiring[†] is an ideal. The potential for noncancellative products was one motivation for Construction 3.5, so the next result becomes relevant.

Proposition 4.5. Suppose z is a ν -noncancellative product, with $\ell = s(z)$. Then $\ell = 2\ell$. In particular, if ℓ is finite, then $\ell = 0$.

Proof. If $^{[\ell]}z = ab \cong_\nu ac$ with $b^\nu > c^\nu$, then

$$^{[\ell]}z = ab = a(b + c) = ab + ac = ^{[\ell]}z + ^{[\ell]}z = ^{[2\ell]}z,$$

implying $\ell = 2\ell$. □

Since 0 and ∞ are multiplicative idempotents of L , one could formulate an analogous definition using the layer at ∞ instead of at 0, and indeed this version is implicit in some of our work on superalgebras and supervaluations, such as [20] and [21]. However, there are several good reasons for using the 0 layer in place of the ∞ layer.

- (1) R_∞ corresponds to the image of the ghost map ν , which may involve considerable contraction. On the other hand, we often do not want any contraction to R_0 .
- (2) In some ways, R_0 and R_∞ should be complements, as indicated presently.
- (3) R_0 is an ideal which behaves much like a zero element. In particular, it is more intuitive for the zero element (if it exists) to belong to R_0 .

(4) Remark 4.10 below formalizes the notion that R_0 also has tangible properties.

Remark 4.6. *The 0-layer and the ∞ -layer behave similarly, since both 0 and ∞ are absorbing elements of L , except that 0 also absorbs ∞ in the sense that $0 \cdot \infty = 0$. In case $\infty \in L$ but $0 \notin L$, R_∞ is an ideal of R that can often be used to replace R_0 in the above discussion.*

One difference between the 0 layer and the ∞ layer is that for $a \cong_\nu b$ with $b \in R_\ell$, if $a \in R_0$ then $s(a + b) = \ell$, whereas if $a \in R_\infty$ then $s(a + b) = \infty$.

Lemma 4.7. *The layer R_0 is also an ideal of R . If furthermore $0_R \in R$, then $0_R \in R_0$.*

Proof. The first assertion is clear. Suppose $0_R \in R_k$. Then for any $a \in R_0$ we have

$$0_R = 0_R \cdot a \in R_{k \cdot 0} = R_0.$$

□

Remark 4.8. *If $\infty \in L$, then R_∞ is a monoid, and $R_0 \cup R_\infty$ is an ideal of R .*

Lemma 4.9. *If \mathcal{M} is any submonoid of a layered semiring † $R := (R, L, s, (\nu_{m,\ell}))$, then the additive sub-semigroup $\overline{\mathcal{M}}$ of R generated by \mathcal{M} is also a layered semiring † .*

Proof. $\overline{\mathcal{M}}$ is closed under multiplication, and thus is a semiring † . Axiom A1 is given, and the other axioms follow a fortiori. □

4.1. The $\{0, 1\}$ -submonoid. Since in general R_1 no longer turns out to be a monoid, we must also take into account the 0-layer.

Remark 4.10. *$R_0 \cup R_1$ is a submonoid of R .*

Definition 4.11. *The $\{0, 1\}$ -submonoid is the submonoid of R generated by R_1 .*

Thus, the $\{0, 1\}$ -submonoid is contained in $R_0 \cup R_1$. Since $1_R \in R_1$, every invertible element of the fundamental submonoid must lie in R_1 .

Proposition 4.12. *Suppose R is an L -layered semiring † . Then \cong_ν is an equivalence relation, whose set \mathcal{G} of equivalence classes is a monoid, which is ordered when R is ν -bipotent. In this case, the $\{0, 1\}$ -submonoid \mathcal{T} of R has an m -valuation $\nu : \mathcal{T} \rightarrow \mathcal{G}$ satisfying $a \mapsto [a^\nu]$.*

Proof. \cong_ν is an equivalence relation by definition, and the equivalence classes comprise a monoid in view of Axiom A3. When R is ν -bipotent, we get an ordered monoid by Remark 4.4, and ν is an m -valuation by Axiom A3. □

We are interested in generation by the $\{0, 1\}$ -submonoid.

Definition 4.13. *The **tangibly generated** sub-semiring † $R_{\langle 1 \rangle}$ of an L -layered semiring † R is the sub-semiring † generated by R_1 ; the semiring † R is **tangibly generated** if $R_{\langle 1 \rangle} = R$.*

Thus, R is tangibly generated if $R = (\bigcup_{k \in L} \nu_{k,1}(R_1)) \cup R_0$. Passing to $R_{\langle 1 \rangle}$ may shrink the sorting set.

Lemma 4.14. *The tangibly generated sub-semiring † $R_{\langle 1 \rangle}$ of a ν -bipotent layered semiring † is a tangibly generated, ν -bipotent layered semiring † with respect to the sorting sub-semiring † of L generated by 1_L . If R is a layered pre-domain † , then $R_{\langle 1 \rangle}$ is a layered pre-domain † whose 0-layer is empty.*

Proof. The axioms are verified a fortiori, since addition only involves adding sorts, starting with 1_L . For the second assertion, since addition cannot lower the sort, we do not get any elements of sort 0. □

Thus, replacing R by its tangibly generated sub-semiring † enables us to assume that $(L, +)$ is generated by 1 and 0.

Example 4.15. *Construction 3.5 is tangibly generated.*

It turns out that we could develop the theory under the weaker condition that L is a partially pre-ordered multiplicative monoid, and we sketch the appropriate changes at the end of the Appendix.

Example 4.16. Given any ideal \mathfrak{a} of an L -layered semiring[†] R , we formally define $R_{\mathfrak{a}}$ to be R with the same semiring[†] operations, and to have the same sort function as R , except that now $s(a) = 0$ for every $a \in \mathfrak{a}$. In other words,

$$(R_{\mathfrak{a}})_0 := R_0 \cup \mathfrak{a}; \quad (R_{\mathfrak{a}})_{\ell} := R_{\ell} \setminus (\mathfrak{a} \cap R_{\ell}).$$

Now define

$$\bar{\mathfrak{a}} := \{b \in R : b \cong_{\nu} a\}.$$

Then $\bar{\mathfrak{a}} \triangleleft R$, so we could use $\bar{\mathfrak{a}}$ instead of \mathfrak{a} .

Proposition 4.17. $R_{\mathfrak{a}}$ is a semiring[†].

Proof. We need to check associativity and distributivity. But this is clear unless we are using elements of \mathfrak{a} , and then associativity holds because all products have layer 0. Likewise, to see that $a(b+c)$ and $ab+ac$ have the same layer, note this is clear if $s(a) = 0$ or if $s(b+c) \neq 0$. Thus we may assume that $s(b+c) = 0$, and again we are done if $s(b) = s(c) = 0$, so we may assume that $s(b) = 0$ and $s(c) \neq 0$ with $b >_{\nu} c$ but $ab \cong_{\nu} ac$. But then

$$s(a(b+c)) = s(ab) + s(ac) = s(ab),$$

so $a(b+c) = ab = ab+ac$. □

Remark 4.18. If $R \setminus \mathfrak{a}$ is finite, then $(R_{\mathfrak{a}})_1$ is a finite set. Thus, we have a way of “contracting” the tangible component to a finite set.

One instance of arithmetic significance is when $R = \mathcal{R}(L, \mathbb{N} \cup \{0\})$ where $\mathfrak{a} = \{^{[\ell]}n : n > q, \ell \in L\}$ for some $q \in \mathbb{N}$. In this case, we can “compress” \mathfrak{a} to a single element in R_0 .

Example 4.19 (The layered ν -truncated semiring[†]). Take an ordered semiring L and ordered triple $(\mathcal{M}, \mathcal{G}, v)$, with $R = \mathcal{R}(L \setminus \{0\}, \mathcal{M})$, and fixing $q > \mathbb{1}_{\mathcal{M}}$ in \mathcal{M} , define

$$\mathfrak{a} := \{^{[\ell]}a : v(a) \geq v(q)\} \triangleleft R.$$

Then $R_{\mathfrak{a}}$ contracts to the L -layered semiring[†]

$$\{^{[k]}a : \ell \in L, a < q\} \cup \{^{[0]}q\},$$

where addition is defined as in Construction 3.5, and multiplication $^{[k]}a \cdot ^{[\ell]}b$ is given as in Equation (3.3) except for $ab \geq q$, in which case $^{[k]}a \cdot ^{[\ell]}b = ^{[0]}q$ for any $k, \ell \in L$. Addition is given by

$$^{[k]}a + ^{[0]}q = ^{[0]}q.$$

The sort transition maps are as in Construction 3.5. Thus, $^{[0]}q$ is the special infinite element.

When instead the layering semiring[†] L is finite, we see that $R_1 \cup \{^{[0]}q\}$ is a finite set, which merits further study using arithmetic techniques.

Here is a way to make L finite.

Example 4.20 (The L -truncated semiring[†]). Take an ordered semiring L and ordered triple $(\mathcal{M}, \mathcal{G}, v)$, with $R = \mathcal{R}(L \setminus \{0\}, \mathcal{M})$, and fix $m > \mathbb{1}_{\mathcal{M}}$ in L . Then $R_{\mathfrak{a}}$ contracts to the L -layered semiring[†]

$$\{^{[\ell]}a : \ell \leq m \in L, a \in \mathcal{M}\},$$

where addition is defined as in Construction 3.5, and multiplication $^{[k]}a \cdot ^{[\ell]}b$ is given as in Equation (3.3) except for $k\ell \geq m$, in which case $^{[k]}a \cdot ^{[\ell]}b = ^{[m]}ab$. Addition is given by

$$^{[k]}a + ^{[m]}0 = ^{[k]}a.$$

The sort transition maps are as in Construction 3.5. Thus, R_m is the special infinite layer.

Thus, the two kinds of truncation can interweave to create finite layered structures.

4.2. The case of onto sort transition maps. We write e_ℓ for $\nu_{\ell,1}(\mathbb{1}_R)$. Here is a key simplification for layered domains[†] when the sort transition maps are onto, which enables us to reduce many results to the tangible case:

Lemma 4.21. *If R is an L -layered semiring[†] and $a \in R_\ell$ with $\nu_{\ell,1} : R_1 \rightarrow R_\ell$ onto, then $a = e_\ell a_1$ for some $a_1 \in R_1$.*

Proof. Taking $a_1 \in R_1$ for which $\nu_{\ell,1}(a_1) = a$, we have $a = \nu_{\ell,1}(a_1) = e_\ell a_1$. \square

Note 4.22. *Lemma 4.21 enables us to simplify the theory for any layer $\ell > 1$ for which $\nu_{\ell,1}$ is onto. When $\ell < 1$ we could go in the opposite direction, and define e_ℓ such that $\nu_{1,\ell}(e_\ell) = \mathbb{1}_R$. This will be well-defined when $\nu_{1,\ell}$ is 1:1 since, writing $\ell = \frac{m}{n}$ for any $a \in R_\ell$ with $\nu_{1,\ell}(a) = \mathbb{1}_R$, we have*

$$ne_\ell = ne_{m/n} = e_m = \nu_{m,\ell}(a) = na, \quad (4.2)$$

implying $a = e_\ell$.

4.3. Adjoining the 0-layer. Starting with an L -layered pre-domain[†] R with respect to a semiring[†] L , we can adjoin a zero layer R_0 formally in several ways. The first way is simply by adjoining a zero element to R .

Remark 4.23. *For any layered pre-domain[†] R with respect to a semiring[†] L , the semiring*

$$R \dot{\cup} \{0_R\}$$

can be layered with respect to the semiring

$$L^0 := L \dot{\cup} \{0_L\},$$

where we take $R_0 := \{0_R\}$, putting it in the zero layer as seen by applying the argument of Proposition 4.17. We take the sort transition maps $\nu_{0,\ell}(a) := 0_R$ for all $\ell \neq 0$ and $a \in R$.

However, this is not the only possibility for the zero layer, as we saw in [17, Remark 3.8].

Construction 4.24. *If R is a uniform L -layered pre-domain[†], where L is a semiring[†], then adjoining $\{0\}$ formally to L as the unique minimal element, we can form a uniform L^0 -layered semiring[†] $R \dot{\cup} R_0$, where $R_0 := e_0 R_1$ is another copy of R_1 , under the same rules of addition and multiplication given by Construction 3.2.*

Proof. If $a = e_0 a_1$, $b = e_k b_1$, and $c = e_\ell c_1$ for $a_1, b_1, c_1 \in R_1$, then

$$(ab)c = e_0 e_k e_\ell (a_1 b_1) c_1 = e_0 e_k e_\ell a_1 (b_1 c_1) = e_0 a_1 (b_1 c_1) = e_0 a_1 (b_1 c_1) = a(bc),$$

yielding associativity of multiplication. To see distributivity, we note that $e_k b_1 + e_\ell c_1 = e_m (b_1 + c_1)$ where $m \in \{k, \ell, k + \ell\}$, so

$$a(b + c) = e_0 e_m a_1 (b_1 + c_1) = e_0 a_1 (b_1 + c_1) = e_0 a_1 b_1 + e_0 a_1 c_1 = e_0 e_k a_1 b_1 + e_0 e_\ell a_1 c_1 = ab + ac.$$

Associativity of addition is similar. Finally, if $a = 0_R \in R_0$ and $b \in R_\ell$, then $ab \in R_{0,\ell} = R_0$. \square

Since we have several ways of adjoining a zero layer, the following observation is useful.

Proposition 4.25. *For any semiring R layered with respect to a semiring[†] L , $R \dot{\cup} \{0_R\}$ is an L^0 -layered sub-semiring of $R \dot{\cup} R_0$.*

More generally, for any ideal \mathfrak{a} of R , writing \mathfrak{a}_0 for $\mathfrak{a} \cap R_0$, we have $(\bigcup_{\ell \neq 0} R_\ell) \dot{\cup} \mathfrak{a}_0$ is an L^0 -layered sub-semiring of $R \dot{\cup} R_0$.

Proof. If $a \in \mathfrak{a}_0$ and $b \in R_\ell$, then $ab \in R_{0,\ell} = R_0$, implying $ab \in \mathfrak{a}_0$. \square

This gives rise to the question of whether we should adjoin the entire 0-layer, or just 0_R ? Although one's experience from classical algebra might lead one to adjoin only 0_R , there are situations in which one might need other elements in R_0 in order to distinguish polynomials.

4.4. Adjoining the absolute ghost layer, and the passage to standard supertropical domains[†]. Even when L originally does not contain an infinite element a priori, L -layered bi-domains[†] tie in directly with the (standard) supertropical theory, via a ghost layer introduced at a new element ∞ which we adjoin. (This works even when (\geq) is merely a partial order on L , although it is easier when (\geq) is a total order.)

Remark 4.26. Any L -layered semiring[†] $(R, L, s, (\nu_{m,\ell}))$ is a directed system with respect to the set L , as described in [23, p. 71]. Hence, by [23, Theorem 2.8], the layers R_k have a direct limit which we denote R_∞ , and maps

$$\nu_{\infty,k} : R_k \rightarrow R_\infty$$

such that $\nu_{\infty,k} = \nu_{\infty,\ell} \circ \nu_{\ell,k}$ for each $a \in R_k$ and all $k < \ell$. Since $R = \bigcup_k R_k$, we can piece together these maps $\nu_{\infty,k}$ to a map $\nu : R \rightarrow R_\infty$. We define

$$e = e_\infty := \nu(1_R), \quad (4.3)$$

easily seen to be the unit element of R_∞ .

We write a^ν for $\nu(a) \in R_\infty$. Thus $a^\nu = b^\nu$ iff $a \cong_\nu b$ in our previous notation.

We call R_∞ the **absolute ghost layer** and ν the (absolute) **ghost map** of R . Note that in the uniform case, R_∞ is just another copy of R_1 , so we can dispense with direct limits.

Theorem 4.27. Suppose $R := (R, L, s, (\nu_{m,\ell}))$ is an L -layered semiring[†]. Then the absolute ghost layer R_∞ is a bipotent semiring[†]. The ghost map $\nu : R \rightarrow R_\infty$ is a semiring[†] homomorphism. Define

$$U = U(R) := R \dot{\cup} R_\infty.$$

Then U is a semiring[†] under the given operations of R and R_∞ , together with

$$\begin{aligned} a \cdot b^\nu &:= (ab)^\nu; \\ a + b^\nu &:= \begin{cases} a & \text{if } ea > eb, \\ b^\nu & \text{if } ea \leq eb. \end{cases} \end{aligned}$$

Also, extend ν to a map $\nu_U : U \rightarrow R_\infty$ by taking ν_U to be the identity on R_∞ . Then U has ghost ideal $\mathcal{G} = \mathcal{G}(U) := R_\infty$, in the sense of [21], and $\nu_U(a) = ea$ for every a in U .

Then U can be modified to a supertropical semiring[†]

$$\mathcal{R}_{1,\infty} := R_1 \dot{\cup} \mathcal{G},$$

retaining the given multiplication \cdot of U , but with new addition \oplus given by the rules

$$a \oplus b := \begin{cases} a & \text{if } ea > eb, \\ b & \text{if } ea < eb, \\ ea & \text{if } ea = eb. \end{cases} \quad (4.4)$$

Proof. Axiom A3 tells us that

$$\nu_{mm',k\ell}(a \cdot b) = \nu_{m,k}(a) \cdot \nu_{m',\ell}(b)$$

for any $a \in R_k$ and $b \in R_\ell$; taking limits yields

$$\nu(a \cdot b) = \nu(a) \cdot \nu(b).$$

Likewise, Axiom B tells us that

$$\nu(a + b) = \nu(a) + \nu(b).$$

The other verifications are also easy. By (4.3) we have

$$\nu(x) = e \cdot x \quad \text{for every } x \in R.$$

Thus $\nu \circ \nu = \nu$, and also $\nu : R \rightarrow \mathcal{G}$ is a semiring[†] homomorphism from R onto $\mathcal{G} = \mathcal{G}(U)$.

We extend the ν -equivalence relation from R to U by decreeing that $a \equiv_U b$ iff a and b have the same value under ν .

We turn to the last assertion. Due to (4.4) we have

$$a \oplus b = a + b \quad \text{if } a \not\equiv_\nu b.$$

On the other hand,

$$a \oplus b = e(a + b) \quad \text{if} \quad a \cong_\nu b.$$

Note that

$$a \oplus b \cong_\nu a + b$$

in all cases. Also, $\mathcal{G}(U) := R_\infty = \mathcal{G}(\mathcal{R}_{1,\infty})$. \square

We may regard $\mathcal{R}_{1,\infty} := (\mathcal{R}_{1,\infty}, \oplus, \cdot)$ as a degeneration of the semiring[†] $U := U(R)$, where all the ghost layers have been coalesced to R_∞ . When $L = L_{\geq 1}$, then there is a semiring[†] homomorphism $U \rightarrow \mathcal{R}_{1,\infty}$ given by

$$a \mapsto \begin{cases} a & \text{for } a \in R_1 \cup R_\infty, \\ \nu(a) & \text{otherwise.} \end{cases}$$

We are now in a position to see why Construction 3.2 of a uniform L -pre-domain[†] is generic. We recall

$$R(L, \mathcal{G}) := \{ {}^{[\ell]}a \mid a \in \mathcal{G}, \ell \in L \}.$$

Remark 4.28. In a uniform L -layered pre-domain[†], we can define $\nu_{k,\ell}$ for $0 < k < \ell$ to be $\nu_{\ell,k}^{-1}$. Thus, $\nu_{k,\ell}$ is defined for all $0 < k, \ell \in L$.

5. MORPHISMS OF LAYERED SEMIRINGS[†]

In order to understand layered categories, we need a good notion of morphism. This is easiest to describe for layered domains[†].

5.1. Layered homomorphisms. We assume that L is non-negative.

Definition 5.1. A *layered homomorphism*

$$(\varphi, \rho) : (R, L, s, (\nu_{m,\ell})) \rightarrow (R', L', s', (\nu'_{m,\ell}))$$

of uniform L -layered pre-domains[†] is a semiring[†] homomorphism $\rho : L \rightarrow L'$ preserving the given partial orders, i.e., satisfying the condition:

$$\text{M1. } k \leq \ell \text{ implies } \rho(k) \leq \rho(\ell).$$

together with a semiring[†] homomorphism $\varphi : R \rightarrow R'$ such that

$$\text{M2. } s'(\varphi(a)) \geq \rho(s(a)), \quad \forall a \in R.$$

The definition becomes more complicated when $0 \in L$; then we need to modify Axiom M2 to:

$$\text{M2'. } s'(\varphi(a)) = \ell, \text{ where } \ell = 0 \text{ or } \ell \geq \rho(s(a)), \quad \forall a \in R.$$

We always write $\Phi := (\varphi, \rho)$. We often assume $L = L'$ and $\rho = \text{id}_L$; we call Φ a **natural homomorphism** in this situation.

Lemma 5.2. Write $e_{\ell,R}$ for e_ℓ in R . Then $\varphi(e_{\ell,R}) = e_{\ell,R'}$, for each ℓ in the sub-semiring[†] of L (resp. L') generated by 1.

Proof. Then $\varphi(e_{1,R}) = \varphi(\mathbb{1}_R) = \mathbb{1}_{R'} = e_{1,R'}$. Thus, for each $n \in \mathbb{N}$, we have

$$\varphi(e_{n,R}) = \varphi(e_{1,R} + \cdots + e_{1,R}) = \varphi(e_{1,R}) + \cdots + \varphi(e_{1,R}) = e_{1,R'} + \cdots + e_{1,R'} = e_{n,R'}.$$

\square

It follows at once that the homomorphism φ is given by its action on R_1 .

Proposition 5.3. If $a = e_\ell a_1$ as in Lemma 4.21, then

$$\varphi(a) = \varphi(e_{\ell,R})\varphi(a_1) = e_{\ell,R'}\varphi(a_1), \quad \forall \ell > 0. \quad (5.1)$$

Proof. $\varphi(a) = \varphi(e_{\ell,R})\varphi(a_1) = e_{\ell,R'}\varphi(a_1)$. \square

Corollary 5.4. Equation (5.1) holds automatically whenever R is uniform L -layered.

Proof. Lemma 4.21 is applicable. \square

Proposition 5.5. *Suppose $\varphi : R \rightarrow R'$ is a layered homomorphism, and R is tangibly generated. Then φ is determined by its action on $R_0 \cup R_1$, via the formula*

$$\varphi\left(\sum_i a_i\right) = \sum_i \varphi(a_i).$$

Proof. It is enough to check sums, in view of Lemma 4.9. \square

Our category will be comprised of the tangibly generated L -layered semirings[†].

5.2. Layered supervvaluations and the layered analytification. In case our layered semiring[†] is not uniform, we need a more general notion of morphism, treated in [19]. To understand what is going on, we need to generalize the notion of “valuation.” Valuations are important in algebraic geometry, and play a key role in tropical theory largely because of the following example.

Example 5.6. *Recall that the field \mathbb{K} of Puiseux series over an algebraically closed field F is defined to be the set of all series of the form*

$$p(t) := \sum_{\tau \in T} c_\tau t^\tau, \quad c_\tau \in F, \quad (5.2)$$

with $T \subset \mathbb{Q}$ well-ordered and bounded from below, endowed with the valuation $\text{Val} : \mathbb{K}^\times \rightarrow \mathbb{R}$ given by

$$\text{Val}(p(t)) := -\min\{\tau \in T : c_\tau \neq 0_F\}, \quad p(t) \in \mathbb{K} \setminus \{0_{\mathbb{K}}\}. \quad (5.3)$$

A word about notation: Given a valuation (or, more generally, an m-valuation) v , one can replace v by $-v$ and reverse the customary inequality to get

$$v(a + b) \leq \max\{v(a), v(b)\},$$

which is more compatible with the max-plus set-up.

Payne [28] has developed an algebraic version of Berkovich’s theory of analytification, which can be viewed as the limit of tropicalizations. In his theory, a **multiplicative seminorm** $|\cdot| : W \rightarrow \mathbb{R}$ on a ring W is a multiplicative map satisfying the triangle inequality

$$|a + b| \leq |a| + |b|.$$

The underlying space in Payne [27] is the set of multiplicative seminorms from $K[\lambda_1, \dots, \lambda_n]$ to $\mathbb{R}_{>0}$ extending v , for a given m-valuation $v : K \rightarrow \mathbb{R}_{>0}$. We generalize this definition by taking an arbitrary ordered semiring[†] instead of $\mathbb{R}_{>0}$.

The supertropical version, the **strong supervvaluation**, is defined in [16, Definition 4.1 and Definition 9.9] as a monoid homomorphism φ satisfying $\varphi(a) + \varphi(b) \underset{\text{gs}}{=} \varphi(a + b)$, where $\underset{\text{gs}}{=}$ is the ghost surpassing relation of [16, Definition 9.1]. In this way, strong supervvaluations generalize seminorms.

Here is the layered analog.

Definition 5.7. *A **layered supervvaluation** on a ring W is a map $\varphi : W \rightarrow R$ from W to an L -layered semiring R with the following properties:*

$$\begin{aligned} LV1 : & \varphi(1) = \mathbb{1}_R, \\ LV2 : & \forall a, b \in R : \varphi(ab) = \varphi(a)\varphi(b), \\ LV3 : & \forall a, b \in R : \varphi(a + b) \leq_\nu \varphi(a) + \varphi(b), \\ LV4 : & \varphi(0) = 0_R. \end{aligned}$$

A $\{0, 1\}$ -**layered supervvaluation** on a ring W is a layered supervvaluation $\Phi : W^\times \rightarrow R$, where $W^\times := W \setminus \{0\}$, such that $\Phi(W) \subseteq R_0 \cup R_1$.

Proposition 5.8. *Suppose that $R := \mathcal{R}(L, \mathcal{G})$ an L -layered bi-domain[†]. If $\Phi : W \rightarrow \mathcal{G}$ is a $\{0, 1\}$ -layered supervvaluation of a ring W , then $\Phi(a)$ is tangible for every invertible element w of W . (In particular, if W is a field, then $\Phi(W^\times)$ is tangible.)*

Proof. $\Phi(w)\Phi(w^{-1}) = \Phi(1) = \mathbb{1}_R$, so $\Phi(w) \notin R_0$, and thus is tangible. \square

In this situation, the tangible layer determines the layered supervaluation.

The morphisms in the layered category should then be those maps which transfer one layered supervaluation to another. In the standard supertropical situation, these are the transmissions of [18], which are given in the layered setting in [19]. This paves the way for the following concept, with, notation as in Example 5.6:

Remark 5.9. Let $R := \mathcal{R}(L, \mathcal{G})$, and view Val as the composite map of monoids

$$\mathbb{K} \xrightarrow{\text{Val}} \mathcal{G} \cong R_1 \subseteq R.$$

Then for any affine algebraic variety X over \mathbb{K} , the space of $\{0, 1\}$ -layered valuations from $\mathbb{K}[\lambda_1, \dots, \lambda_n]$ to R extends \mathbb{K}^{an} of [28], and its theory invites further study.

5.3. Surpassing and surpassed maps. In line with the philosophy of this paper, we would like to introduce the category of L -layered semirings[†]. Having the layered semirings[†] in hand, we next need to define the relevant morphisms. From now on, to avoid complications, we assume that R is a uniform, L -layered pre-domain[†]. As indicated in the introduction, although the natural definition from the context of semirings[†] is good enough for most purposes, a sophisticated analysis requires us to consider the notion of “supervaluation,” and how this would relate to morphisms that preserve the properties of supervaluations, which we will discuss in §7. But a more naive approach suits our needs in many situations.

5.3.1. The surpassing relation. For $\ell \in L$, an ℓ -ghost sort is an element of the form $\ell + k$, for positive $k \in L$. We say a is ℓ -ghost if $s(a)$ is an ℓ -ghost sort. Note that the infinite element ∞ of L is a “self-ghost sort,” in the sense that $\infty + m = \infty$ implies that ∞ is an ∞ -ghost sort.

Here is a key relation in the theory.

Definition 5.10. The *surpassing L -relation* \models_L is given by

$$a \models_L b \quad \text{iff either} \quad \begin{cases} a = b + c & \text{with } c \text{ } s(b)\text{-ghost,} \\ a = b, \\ a \cong_\nu b & \text{with } a \text{ } s(b)\text{-ghost.} \end{cases} \quad (5.4)$$

It follows that if $a \models_L b$, then $a + b$ is $s(b)$ -ghost. When $a \neq b$, this means $a \geq_\nu b$ and a is $s(b)$ -ghost.

Definition 5.11. The *surpassing (L, ν) -relation* \models_ν is given by

$$a \models_\nu b \quad \text{iff} \quad a \models_L b \text{ and } a \cong_\nu b. \quad (5.5)$$

Lemma 5.12. The surpassing L -relation \models_L and the surpassing (L, ν) -relation \models_ν are half-congruences.

Proof. Pointwise verifications. \square

Remark 5.13. The congruence Ω defined by declaring $a \equiv b$ when $a \models_\nu b$ and $b \models_\nu a$, yields an ordered monoid.

5.3.2. Surpassing morphisms. We also weaken the notion of layered homomorphism for layered semirings[†].

Definition 5.14. A *surpassing map* $\varphi : R \rightarrow R'$ is a (multiplicative) monoid homomorphism such that $\varphi(a + b) \models_\nu \varphi(a) + \varphi(b)$.

A *surpassed map* $\varphi : R \rightarrow R'$ is a monoid homomorphism such that $\varphi(a) + \varphi(b) \models_\nu \varphi(a + b)$.

A *0-excepted homomorphism* $\varphi : R \rightarrow R'$ is a monoid homomorphism such that $\varphi(a) + \varphi(b) = \varphi(a + b)$ whenever $s(a), s(b) > 0$.

(In other words, a 0-excepted homomorphism could fail to be a semiring[†] homomorphism only because of the behavior of the 0 sort.)

We write $R_{\geq \ell}$ for $\cup_{k \geq \ell} R_k$.

Example 5.15. If $a >_\nu b$, then $(a + b)^m = a^m$. Hence, for any given m , the Frobenius property

$$(a + b)^m \stackrel{L}{\equiv}_\nu a^m + b^m \quad (5.6)$$

from [17, Remark 5.26] is satisfied in any L -layered semiring[†] and, the Frobenius map $a \mapsto a^m$ is a surpassing map in $R_{\geq 1} \cup R_0$.

Proposition 5.16. We have the surpassing map $\varphi : M_n(R) \rightarrow M_n(R)$ given by $(a_{i,j}) \mapsto (a_{i,j}^m)$.

Proof. We need to show that $(c_{i,j}^m) = (a_{i,j}^m)(b_{i,j}^m)$, where $c_{i,j} = \sum_k a_{i,k} b_{k,j}$. But by (5.6),

$$c_{i,j}^m = \left(\sum_k a_{i,k} b_{k,j} \right)^m \stackrel{L}{\equiv} \sum_k (a_{i,k} b_{k,j})^m = \sum_k a_{i,k}^m b_{k,j}^m.$$

□

Example 5.17. In the standard supertropical situation, the supertropical determinant (i.e., the permanent) is a surpassing map, by [22].

Proposition 5.18. Any surpassing map φ preserves ν , in the sense that if $a \geq_\nu b$, then $\varphi(a) \geq_\nu \varphi(b)$.

Proof. $\varphi(a) \stackrel{L}{\equiv}_\nu \varphi(a + b) \stackrel{L}{\equiv}_\nu \varphi(a) + \varphi(b)$, implying $\varphi(a) \geq_\nu \varphi(b)$. □

Nevertheless, we take the morphisms in this category to be the 0-excepted homomorphisms.

5.4. Layered morphisms. Since morphisms lie at the heart of category theory, the time has come to consider the morphisms that arise for layered semirings[†].

Definition 5.19. A **layered morphism** of L -layered semirings[†] is a map

$$\Phi := (\varphi, \rho) : (R, L, s, (\nu_{m,\ell})) \rightarrow (R', L', s', (\nu'_{m',\ell'})) \quad (5.7)$$

where $\rho : L \rightarrow L'$ is a semiring[†] homomorphism, together with a 0-excepted homomorphism $\varphi : R \rightarrow R'$ such that

- M1. $s'(\varphi(a)) \geq \rho(s(a))$ or $s'(\varphi(a)) = 0$.
- M2. For all $a \in R_k$, $s(\varphi(\nu_{\ell,k})(a)) \stackrel{L'}{\equiv}_\nu s(\varphi(a))$ for all $\ell \geq k$.
- M3. If $a \stackrel{L}{\equiv}_\nu b$, then $\varphi(a) \stackrel{L'}{\equiv}_\nu \varphi(b)$ (taken in the context of the $\nu'_{m',\ell'}$).

A **layered homomorphism** is a layered morphism such that $\varphi : R \rightarrow R'$ is a semiring[†] homomorphism.

We always write $\Phi := (\varphi, \rho) : (R, L, s, (\nu_{m,\ell})) \rightarrow (R', L', s', (\nu'_{m',\ell'}))$, denoted as $\Phi : R \rightarrow R'$ when unambiguous. In most of the following examples, the sorting semirings[†] L and L' are the same.

Example 5.20. Here are some examples of layered homomorphisms. We assume throughout that R is an L -layered semiring[†], although sometimes we consider the role of $\mathbb{0}_R$ if it exists.

- (a) In the max-plus situation, when $L = \{1\}$, ρ must be the identity, and Φ is just a semiring[†] homomorphism. When $L = \{0, 1\}$ and $R_0 = \{\mathbb{0}_R\}$, we must have $\varphi(\mathbb{0}_R) = \mathbb{0}_R$.
- (b) In the “standard supertropical situation without 0,” when $L = \{1, \infty\}$, $\Phi(R_\infty) = R_\infty$.
- (c) In the “standard supertropical situation with 0,” when $L = \{0, 1, \infty\}$, and $R_0 = \{\mathbb{0}_R\}$, Φ must send the ghost layer R_∞ to $R_\infty \cup R_0$. If $\mathfrak{a} \triangleleft R$ and $\mathfrak{a} \supset R_\infty$, one could take $R' = R$ as a set, with $R'_1 = R_1 \setminus \mathfrak{a}$ and $R'_0 = \mathfrak{a}$. The identity map is clearly a layered homomorphism; its application “expands the zero level” to \mathfrak{a} .
- (d) Notation as in Theorem 3.6, we define a layered homomorphism $\mathcal{R}(L, \mathcal{G}) \rightarrow \mathcal{R}(L, \mathcal{G})_{\mathfrak{a}}$ given by the identity map on all elements of $\mathcal{R}(L, \mathcal{G}) \setminus \mathfrak{a}$, and $^{[\ell]}a \mapsto ^{[0]}a$ for every $a \in \mathfrak{a}$.
- (e) Any semiring[†] homomorphism $\rho : L \rightarrow L'$ induces a layered homomorphism $\mathcal{R}(L, \mathcal{G}) \rightarrow \mathcal{R}(L', \mathcal{G})$ given by $^{[\ell]}a \mapsto ^{[\rho(\ell)]}a$.
- (f) The natural injections $R_{\geq 1} \cup R_0 \rightarrow R$ and $\{\bigcup_\ell R_\ell : \ell \in \mathbb{N}\} \rightarrow R$ are both examples of layered homomorphisms.
- (g) The truncation maps of Example 4.19 and Example 4.20 are layered homomorphisms.

- (h) Suppose $\mathfrak{a} \triangleleft R$ is a ν -“upper” ideal in $R_{\geq 1}$ or in $R_{\geq 1} \cup R_0$, by which we mean an ideal of the form $\{r : r \geq_\nu a\}$ or $\{r : r >_\nu a\}$. We define the congruence $\Omega_{\mathfrak{a}}$ on $R_{\mathfrak{a}}$ to be $(\mathfrak{a} \times \mathfrak{a}) \cup \text{diag}(R)$; in other words, $b_1 \equiv_{\mathfrak{a}} b_2$ if $b_1 \cong_\nu b_2$ or if $b_1, b_2 \in \mathfrak{a}$. Then $R_{\mathfrak{a}}/\Omega_{\mathfrak{a}}$ is a layered semiring[†], under the induced multiplication and addition of equivalence classes, and $a \mapsto [a]$ defines a layered homomorphism. Note that all elements of \mathfrak{a} collapse to a single element, as in the Rees quotient construction for semigroups.

Having these examples in hand, one might wonder why we bother with 0-excepted homomorphisms in the definition of morphism. This is in order to make Theorem 6.3 possible.

Proposition 5.21. *Any layered morphism φ on a tangibly generated layered semiring[†] is determined by its action on the tangible submonoid $R_0 \cup R_1$.*

Proof. Since $\varphi(e_k a) = \varphi(e_k) \varphi(a)$, it suffices to check that $\varphi(e_k)$ is uniquely defined. Write

$$e'_k = \varphi(\mathbb{1}_R) + \cdots + \varphi(\mathbb{1}_R),$$

taken k times, whose sort is k . Since $\varphi(\mathbb{1}_R) = \mathbb{1}_{R'}$, we have $e'_k \stackrel{L}{\equiv}_\nu \varphi(e_k)$ by definition of 0-excepted homomorphism. Hence, $s(\varphi(e_k)) \leq k$. But $s(\varphi(e_k)) \geq k$ by Condition M1, implying $s(\varphi(e_k)) = k$, and thus $e'_k = \varphi(e_k)$, as desired. \square

6. THE LAYERED CATEGORIES AND THEIR TROPICALIZATION FUNCTORS

Having assembled the basic concepts, we are finally ready to tie these ideas to tropicalization, by introducing the layered categories. Our objective in this section is to introduce the functor that passes from the “classical algebraic world” of integral domains with valuation to the “layered world,” taking the cue from [21, Definition 2.1], which we recall and restate more formally.

6.1. Identifications of categories of valued monoids and layered semirings[†]. Here is our main layered category.

Definition 6.1. *LaySemi[†] is the category whose objects are tangibly generated layered semirings[†] and whose morphisms are layered morphisms.*

Remark 6.2. *In view of Theorem 3.6 we can define the forgetful functor $\text{LaySemi}^\dagger \rightarrow \text{OMon}^+$ given by sending the L -layered semiring[†] R to $R_0 \cup R_1$.*

Thus, any layered homomorphism yields a homomorphism of the underlying monoid of tangible elements, thereby indicating an identification between categories arising from the construction of layered pre-domains[†] from ordered monoids (and more generally, of layered semirings[†] from valued monoids). But to get the other direction, we need to permit morphisms merely to be surpassed maps, as previously defined.

Theorem 6.3. *For any valued semiring L , there is a faithful **layering functor** $\mathcal{F} : \text{ValMon} \rightarrow \text{LaySemi}^\dagger$, given by sending \mathcal{M} to $\mathcal{R}(L, \mathcal{M})_{\bar{\mathfrak{a}}}$, where \mathfrak{a} is the monoid ideal of noncancellative products, and the ordered homomorphism $\varphi : \mathcal{M} \rightarrow \mathcal{M}'$ to the layered homomorphism $\mathcal{F}\varphi : \mathcal{R}(L, \mathcal{M}) \rightarrow \mathcal{R}(L, \mathcal{M}')$ obtained from φ as follows:*

$\mathcal{F}\varphi$ is defined on $R_0 \cup R_1$ via $\mathcal{F}\varphi([^\ell a]) = [^{\ell'} \varphi(a)]$, where $\ell' = 1$ unless $\varphi(a)$ is a noncancellative product in \mathcal{G}' , in which case $\ell' = 0$.

The functor \mathcal{F} is a left retract of the forgetful function of Remark 6.2.

Proof. The image of an ordered monoid \mathcal{G} is a layered semiring[†], in view of Proposition 4.12, and one sees easily that $\mathcal{F}\varphi$ is a layered morphism since, for $a \geq_\nu b$,

$$\mathcal{F}\varphi([^k a] + [^\ell b]) \cong_\nu \mathcal{F}\varphi([^k a]) \cong_\nu \varphi([^k a]) \cong_\nu \varphi([^k a]) + \varphi([^\ell b]),$$

and $s'(\mathcal{F}\varphi([^k a]))$ is k or 0.

One needs to verify that $\bar{\mathfrak{a}}R_1 \subseteq \bar{\mathfrak{a}}$. But $\mathfrak{a}R_1 \subseteq \mathfrak{a}$ is clear by definition of noncancellative product, yielding $\bar{\mathfrak{a}}R_1 \subseteq \bar{\mathfrak{a}}$.

The morphisms match. The functor \mathcal{F} is faithful, since one recovers the original objects and morphisms by applying the forgetful functor of Remark 6.2. \square

6.1.1. *The layered tropicalization functor.*

Definition 6.4. Given a semiring[†] L , the L -**tropicalization functor**

$$\mathcal{F}_{\text{LTrop}} : \text{ValMon} \longrightarrow \text{LaySemi}^\dagger$$

from the category of valued monoids to the category of uniform layered semirings[†] is defined as follows: $\mathcal{F}_{\text{LTrop}} : (\mathcal{M}, \mathcal{G}, v) \mapsto \mathcal{R}(L, \mathcal{G})_{\mathbf{a}}$ and $\mathcal{F}_{\text{LTrop}} : \phi \mapsto \alpha_\phi$, where \mathbf{a} is the ideal of noncancellative elements of the monoid \mathcal{G} , and, given a morphism $\phi : (\mathcal{M}, \mathcal{G}, v) \rightarrow (\mathcal{M}', \mathcal{G}', v')$ we define $\alpha_\phi : \mathcal{R}(L, \mathcal{G}) \rightarrow \mathcal{R}(L', \mathcal{G}')$, by

$$\alpha_\phi([^\ell]a) := [^k]\phi(a), \quad a \in \mathcal{G}, \quad (6.1)$$

where $k = 0$ if $\phi(a)$ is noncancellative and $k = \ell$ if $\phi(a)$ is cancellative, cf. Formula (2.5).

Note that the L -tropicalization functor $\mathcal{F}_{\text{LTrop}}$ factors as

$$\text{ValMon} \rightarrow \text{OMon} \rightarrow \text{LaySemi}$$

which restricts to $\text{ValMon}^+ \rightarrow \text{OMon}^+ \rightarrow \text{LaySemi}^\dagger$ of [19].

Suppose $v : W^\times \rightarrow \mathcal{G}$ is a valuation on an integral domain W , where $W^\times := W \setminus \{0_W\}$. Let $\mathcal{M} := W^\times$, a multiplicative monoid. Fix $\ell \in L$; usually $\ell = 1$. The restriction of v to \mathcal{M} , which we denote as ψ_ℓ , can be realized as the map sending \mathcal{M} as a set into the ℓ -layer of $\mathcal{R}(L, \mathcal{G})$, given by $\psi_\ell : a \mapsto [^k]v(a)$, where $k = 0$ if a is a noncancellative product and $k = \ell$ otherwise. This is not a homomorphism of semirings[†], since $a + (-a) = 0_W$ whereas $v(-a) = v(a)$, and thus

$$\psi_\ell(a + (-a)) = \psi_\ell(0_W) = 0_R \neq [^{2\ell}]a = \psi_\ell(a) + \psi_\ell(a) = \psi_\ell(a) + \psi_\ell(-a).$$

But this is exactly where the layered theory acts more categorically than the the max-plus theory.

Proposition 6.5. Suppose W is an integral domain with valuation v , and

$$\psi_\ell : \mathcal{M} \rightarrow \mathcal{R}(L, \mathcal{G})_{\mathbf{a}},$$

is the map just described. If $\sum_i a_i = 0_W$ with each a_i in W^\times , then $s(\sum_i \psi_\ell(a_i)) \geq 2$.

Proof. This is really a reformulation of a standard, elementary fact in valuation theory, in which we recall that $v(0_W)$ is undefined. It is well-known that if $\sum_i a_i = 0_W$ then there exist i_1, i_2, \dots such that $v(a_{i_1}) = v(a_{i_2}) = \dots$ which dominate all other $v(a_i)$, since if a single $v(a_{i_1})$ dominated, we would have $0_W = v(\sum_i a_i) = v(a_{i_1})$, a contradiction. Hence,

$$s\left(\sum_i \psi_\ell(a_i)\right) = s(\psi_\ell(a_{i_1})) + s(\psi_\ell(a_{i_2})) + \dots \geq 1 + 1 + \dots \geq 2.$$

□

Thus, we see that the L -tropicalization functor explains the importance of the “surpassing L -relation.”

6.1.2. *The role of Kapranov’s Lemma.* We are ready to extend the considerations of [19, §8.1]. Since Puiseux series play such an important role in tropical geometry, let us understand them in terms of layers.

Remark 6.6. We start with a triple (F, \mathcal{G}, v) , where F for example may be the algebra of Puiseux series, \mathcal{G} an ordered monoid, and $v : F \rightarrow \mathcal{G}$.

Take the layered semiring[†] $R := \mathcal{R}(L, \mathcal{G})$. Define a **Kapranov map** to be a $\{0, 1\}$ -supervaluation satisfying the property:

$$\tilde{v}(a) + \tilde{v}(b) \models_L \tilde{v}(a + b). \quad (6.2)$$

This is the analog of the iq -supervaluation in [16, Definition 11.12]. By Proposition 6.5, we see that the Kapranov map sends any root of f to a corner root of $\tilde{v}(f)$. This general framework of Kapranov’s lemma encompasses tropicalizations of finite Puiseux series introduced in [17] and [19].

7. LAYERED SUPERVALUATIONS AND TRANSMISSIONS: AN ALTERNATIVE APPROACH TO MORPHISMS

In this section we delve deeper into the nature of morphisms, towards what would be the “correct” general definition in the category of layered semirings[†], paralleling the general theory of m -valuations given in [17]. The outcome is somewhat technical, but enables us to define a functor from the functions in the algebraic world to the category of layered function semirings[†], and indicates that Payne’s methods [27] should also be applicable in the layered theory.

In Corollary 7.11, we will see that this approach reduces to Section 6 in many cases.

Since valuations play such an important role, we would like to extend our definition of morphism to include all maps preserving valuations. This route leads us to a layered version of supervaluations and transmissions. See [16], [18], [20] for further details in the supertropical case.

Definition 7.1. *An L -layered supervaluation on a ring W , with respect to a semiring L , is a map $\tilde{v} : W \rightarrow R$ from W to an L -layered semiring R satisfying the following properties.*

$$\begin{aligned} \text{LV1} : \tilde{v}(\mathbb{1}_W) &= \mathbb{1}_R, \\ \text{LV2} : \forall a, b \in R : \tilde{v}(ab) &= \tilde{v}(a)\tilde{v}(b), \\ \text{LV3} : \forall a, b \in R : \tilde{v}(a+b) &\leq_\nu \tilde{v}(a) + \tilde{v}(b), \\ \text{LV4} : \tilde{v}(\mathbb{0}_W) &= \mathbb{0}_R. \end{aligned}$$

A $\{0, 1\}$ -supervaluation on a ring W is an L -layered supervaluation $\tilde{v} : W \rightarrow R$ such that $\tilde{v}(W) \subseteq R_0 \cup R_1$.

An L -layered supervaluation[†] on an integral domain W , with respect to a semiring[†] L , is a map $\tilde{v} : W^\times \rightarrow R$ from $W^\times := W \setminus \{\mathbb{0}_W\}$ to an L -layered pre-domain[†] R with the following properties.

$$\begin{aligned} \text{LV1}^\dagger : \tilde{v}(\mathbb{1}_W) &= \mathbb{1}_R, \\ \text{LV2}^\dagger : \forall a, b \in R : \tilde{v}(ab) &= \tilde{v}(a)\tilde{v}(b), \\ \text{LV3}^\dagger : \forall a, b \in R : \tilde{v}(a+b) &\leq_\nu \tilde{v}(a) + \tilde{v}(b). \end{aligned}$$

To encompass the results of [16] and [18], instead of using layered homomorphisms for our morphisms, we need to consider a “transmissive” property analogous to the one given in [18, Definition 4.3].

Definition 7.2. *If $\tilde{v} : W \rightarrow R$ and $\tilde{w} : W \rightarrow R'$ are L -layered supervaluations, where R has sorting map $s : R \rightarrow L$ and R' has sorting map $s' : R' \rightarrow L$, we say that \tilde{v} **dominates** \tilde{w} if the following properties hold for any $a, b \in W$:*

$$\begin{aligned} \text{D1.} \quad \tilde{v}(a) = \tilde{v}(b) &\Rightarrow \tilde{w}(a) = \tilde{w}(b), \\ \text{D2.} \quad \tilde{v}(a) \leq_\nu \tilde{v}(b) &\Rightarrow \tilde{w}(a) \leq_\nu \tilde{w}(b), \\ \text{D3.} \quad \tilde{v}(a) \in R_0 &\Rightarrow \tilde{w}(a) \in R'_0, \\ \text{D4.} \quad s(\tilde{v}(a)) \leq s'(\tilde{w}(a)) &\text{ whenever } \tilde{w}(a) \notin R'_0. \end{aligned}$$

(We omit D3 and the condition in D4 for layered supervaluations[†], since we do not need to bother with the 0 layer.)

Definition 7.3. *For L -layered domains[†] R and R' and $\mathcal{M} \subset R$, a map $\alpha : \mathcal{M} \rightarrow R'$ is ν -preserving if*

$$a \leq_\nu b \text{ implies } \alpha(a) \leq_\nu \alpha(b)$$

for all $a, b \in R$.

Lemma 7.4. *For any ν -preserving map α , if $a \cong_\nu b$, then $\alpha(a) \cong_\nu \alpha(b)$ for $a, b \in R$.*

Proof. $a \cong_\nu b$ implies $\alpha(a) \leq_\nu \alpha(b)$ and likewise $\alpha(b) \leq_\nu \alpha(a)$, so $\alpha(a) \cong_\nu \alpha(b)$. □

Lemma 7.5. *Let $\tilde{v} : W \rightarrow R$ and $\tilde{w} : W \rightarrow R'$ be L -layered supervaluations. If \tilde{v} dominates \tilde{w} , then there exists a unique ν -preserving map $\alpha_{\tilde{w}, \tilde{v}} : \tilde{v}(W) \rightarrow R'$ with $\tilde{w} = \alpha_{\tilde{w}, \tilde{v}} \circ \tilde{v}$.*

Proof. By D1 we have a well-defined map $\alpha_{\tilde{w}, \tilde{v}} : \tilde{v}(W) \rightarrow \tilde{w}(W)$ given by $\alpha_{\tilde{w}, \tilde{v}}(\tilde{v}(a)) = \tilde{w}(a)$ for all $a \in W$. Furthermore, if $\tilde{v}(a) \leq_\nu \tilde{v}(b)$, then D2 implies $\tilde{w}(a) \leq_\nu \tilde{w}(b)$, so $\alpha_{\tilde{w}, \tilde{v}}$ is ν -preserving. □

Definition 7.6. For layered semirings R and R' , a **transmission** from R to R' is a ν -preserving map $\alpha : \mathcal{M} \rightarrow R'$, with \mathcal{M} a multiplicative submonoid of R , satisfying the following axioms:

$$\begin{aligned} \text{TM1 : } & \alpha(1_R) = 1_{R'}, \\ \text{TM2 : } & \alpha(ab) = \alpha(a)\alpha(b), \quad \forall a, b \in R, \\ \text{TM3 : } & \alpha(a + b) \cong_\nu \alpha(a) + \alpha(b), \quad \text{whenever } a, b, a+b \in \mathcal{M}. \end{aligned}$$

Axioms TM1 and TM2 imply that α is a monoid homomorphism, which we denote as $\alpha : (R, \mathcal{M}) \rightarrow R'$ to emphasize that \mathcal{M} is a submonoid of R . We write \mathcal{M}_ℓ for $R_\ell \cap \mathcal{M}$. A $\{0, 1\}$ -**transmission** from R to R' is a transmission $\alpha : (R, \mathcal{M}) \rightarrow R'$ for which $\alpha(\mathcal{M}_1) \subseteq R'_1 \cup R'_0$.

Lemma 7.7. Axiom TM3 is equivalent to the map α being ν -preserving.

Proof. (\Rightarrow) If $a \leq_\nu b$, then

$$\alpha(b) \cong_\nu \alpha(a + b) \cong_\nu \alpha(a) + \alpha(b),$$

implying $\alpha(a) \leq_\nu \alpha(b)$.

(\Leftarrow) We may assume that $a \leq_\nu b$, implying $a + b \cong_\nu b$. Then $\alpha(a) \leq_\nu \alpha(b)$, so

$$\alpha(a) + \alpha(b) \cong_\nu \alpha(b) \cong_\nu \alpha(a + b).$$

□

Note that the condition of the lemma does not refer explicitly to calculating sums in \mathcal{M} , so we can study transmissions without worrying about addition on \mathcal{M} .

Theorem 7.8. Let $\tilde{v} : W \rightarrow R$ be an L -layered supervaluation and $\tilde{w} : W \rightarrow R'$ an L -layered supervaluation dominated by \tilde{v} . The map $\alpha := \alpha_{\tilde{w}, \tilde{v}} : (R, \tilde{v}(W)) \rightarrow R'$ is a transmission from R to R' .

Conversely, assume that $\tilde{v} : W \rightarrow R$ is an L -layered supervaluation and $\alpha : \tilde{v}(W) \rightarrow R'$ is a transmission from R to an L -layered semiring[†] R' . Then $\alpha \circ \tilde{v} : W \rightarrow R'$ is an L -layered supervaluation dominated by \tilde{v} .

Proof. TM1 and TM2 are obtained from the construction of $\alpha_{\tilde{w}, \tilde{v}}$ in the proof of Lemma 7.5. Now assume that $a \leq_\nu b$, so $\tilde{v}(a) \leq_\nu \tilde{v}(b)$, and thus $\tilde{v}(a) + \tilde{v}(b) \cong_\nu \tilde{v}(b)$. But $\tilde{w}(a) \leq_\nu \tilde{w}(b)$ by D2, so

$$\alpha(\tilde{v}(a)) + \alpha(\tilde{v}(b)) = \tilde{w}(a) + \tilde{w}(b) \cong_\nu \tilde{w}(b) = \alpha(\tilde{v}(b)) \cong_\nu \alpha(\tilde{v}(a) + \tilde{v}(b)).$$

This is TM3.

For the reverse direction, let $\tilde{w} := \alpha \circ \tilde{v}$. Clearly \tilde{w} inherits the properties LV1–LV3 from \tilde{v} , since α satisfies TM1–TM3. □

Corollary 7.9. Every transmission of Theorem 7.8 is ν -preserving.

Proof. α is the map of Lemma 7.5, so is ν -preserving. □

It is evident that every semiring[†] homomorphism from R to R' is a transmission, but there exist transmissions that are not semiring[†] homomorphisms; cf. [16, §9]. Nevertheless, we do get semiring[†] homomorphisms in the following basic case. We say that the transmission α is **homomorphic** if it satisfies the condition

$$\alpha(a + b) = \alpha(a) + \alpha(b) \tag{7.1}$$

whenever $a, b, a + b \in \mathcal{M}$.

Every homomorphic transmission satisfying $\mathcal{M} = R$ is a layered homomorphism, by definition. We say that a ν -preserving map α is **strictly ν -preserving** if $a <_\nu b$ implies that either $\alpha(a) \in R'_0$ or $\alpha(a) <_{\nu'} \alpha(b)$.

Theorem 7.10. Let $\tilde{v} : W \rightarrow R$ be an $\{0, 1\}$ -layered supervaluation and $\tilde{w} : W \rightarrow R'$ an $\{0, 1\}$ -layered supervaluation dominated by \tilde{v} . Then the $\{0, 1\}$ -transmission $\alpha := \alpha_{\tilde{w}, \tilde{v}} : (R, \tilde{v}(W)) \rightarrow R'$ is homomorphic, iff it is strictly ν -preserving.

Proof. (\Rightarrow) Follows from Corollary 7.9.

(\Leftarrow) We need to check (7.1). If $a <_\nu b$, then $\alpha(a + b) = \alpha(b)$, so (7.1) holds iff $\alpha(a) <_\nu \alpha(b)$ or $\alpha(a) \in R_0$. The symmetric argument holds when $b <_\nu a$. Finally, if $a \cong_\nu b$, with $a \in R_0$, then $\alpha(a) \in R_0$, with $\alpha(a) \cong_\nu \alpha(b)$, so

$$\alpha(a + b) = \alpha(b) = \alpha(a) + \alpha(b).$$

Likewise for $b \in R_0$, so we may assume that $a, b \in R_1$. Then $a + b \in R_2$, so there is nothing to check. \square

Corollary 7.11. *Suppose $\tilde{v} : W \rightarrow R$ is an $\{0, 1\}$ -layered supervaluation such that $\tilde{v}(W)$ strictly generates R , and $\tilde{w} : W \rightarrow R'$ is an $\{0, 1\}$ -layered supervaluation dominated by \tilde{v} . Then the $\{0, 1\}$ -transmission $\alpha := \alpha_{\tilde{w}, \tilde{v}} : (R, \tilde{v}(W)) \rightarrow R'$ extends to a layered homomorphism from R to R' , iff α is strictly ν -preserving.*

In particular, when R is uniform, every $\{0, 1\}$ -transmission yields a layered homomorphism.

Remark 7.12. *Since every transmission is a monoid homomorphism, we have a subcategory L -STROP of the category of monoids and monoid homomorphisms, whose objects are layered semirings $(R, L, s, (\nu_{m, \ell}))$, and whose morphisms are the $\{0, 1\}$ -transmissions. Explicitly, $\{0, 1\}$ -transmissions from R to R' and from R' to R'' are described respectively as transmissions $\alpha : (R, \mathcal{M}) \rightarrow R'$ for which $\alpha(\mathcal{M}_1) \subseteq R'_1 \cup R'_0$ and $\alpha' : (R', \mathcal{M}') \rightarrow R''$ for which $\alpha'(\mathcal{M}'_1) \subseteq R''_1 \cup R''_0$. Their composition can be defined as $\alpha' \circ \alpha$ whenever $\alpha(\mathcal{M}) \subseteq \mathcal{M}'_1$.*

This category closely resembles the category STROP of [18] (but with the subtle difference indicated in Remark 4.6), and encompasses the category from §6.1.

8. APPENDIX: LAYERED MONOIDS

At times we do not want additivity at the 0 level, since the vagaries of cancellation complicate the statements and proofs some of the theorems. But then we must give up addition between R_0 and other levels. At this generality, our next structure is not quite a semiring[†], since distributivity does not hold at the 0-layer, but we copy what we can from Definition 4.1.

Definition 8.1. *Suppose (L, \geq) is a directed, partially pre-ordered semiring[†]. An L -layered monoid*

$$R := (R, L, s, (\nu_{m, \ell})),$$

is a multiplicative monoid R which is a disjoint union of subsets R_ℓ , $\ell \in L$, together with addition defined on R_0 and on $R_{>0} := \bigcup_{\ell > 0} R_\ell$ such that

$$R = \bigcup_{\ell \in L} R_\ell, \tag{8.1}$$

*together with a family of **sort transition maps***

$$\nu_{m, \ell} : R_\ell \rightarrow R_m, \quad \forall m \geq \ell > 0,$$

such that

$$\nu_{\ell, \ell} = \text{id}_{R_\ell}$$

for every $\ell \in L$, and

$$\nu_{m, \ell} \circ \nu_{\ell, k} = \nu_{m, k}, \quad \forall m \geq \ell \geq k,$$

whenever both sides are defined. We also require the axioms A1–A4, and B, given presently, to be satisfied.

We define R_∞ to be the direct limit of the R_ℓ , $\ell > 0$, together with a map $\nu : R_\ell \rightarrow R_\infty$, which extends to a map $\nu : R \rightarrow R_\infty$. We write a^ν for $\nu(a)$.

We write $a \cong_\nu b$ for $b \in R_\ell$, whenever $\nu(a) = \nu(b)$. (For $k, \ell > 0$ this means $\nu_{m, k}(a) = \nu_{m, \ell}(b)$ in R_m for some $m \geq k, \ell$. The notation is used generically, as before.) Similarly, we write $a \leq_\nu b$ if $a \cong_\nu b$ or $\nu_{m, k}(a) + \nu_{m, \ell}(b) = \nu_{m, \ell}(b)$ in R_m for some $m \geq k, \ell$.

The axioms are as follows:

- A1. $1_R \in R_1$.
- A2. If $a \in R_k$ and $b \in R_\ell$, then $ab \in m$ where $m \geq k\ell$ or $m = 0$.
- A2'. If $a \in R_0$ or $b \in R_0$, then $ab \in R_0$.
- A3. The product in R is compatible with sort transition maps: Suppose $a \in R_\ell$, $b \in R_{\ell'}$, with $m \geq \ell$ and $m' \geq \ell'$.
If $ab \in R_{\ell''}$ for $\ell'' \geq \ell'\ell$, then $\nu_{m,\ell}(a) \cdot \nu_{m',\ell'}(b) = \nu_{m'',\ell''}(ab)$ for some $m'' \geq mm'$.
- A4. $\nu_{\ell,k}(a) + \nu_{\ell',k}(a) = \nu_{\ell+\ell',k}(a)$ for all $a \in R_k$ and all $\ell, \ell' \geq k$.
- A5. If $a \in R_k$, $b \in R_\ell$, and $c = a + b \in R_{k'}$, then
- $$\nu_{m,k'}(c) = \nu_{m,k}(a) + \nu_{m,\ell}(b)$$
- for each $m \geq k + \ell$.
- A6. $R_{>0}$ is an additive semigroup and R_0 is an $R_{>0}$ -module, in the sense that R_0 is an additive semigroup together with a multiplication $R_0 \times R_{>0} \rightarrow R_0$ satisfying distributivity and associativity whenever defined.

- B. (Supertropicality) Suppose $a \in R_k$, $b \in R_\ell$, and $a \cong_\nu b$. Then $a + b \in R_{k+\ell}$ with $a + b \cong_\nu a$.

The **sorting map** $s : R \rightarrow L$, is a map that sends every element $a \in R_\ell$ to its sort ℓ .

Example 8.2. Suppose R is a layered pre-domain[†], and define formally R_0 to be another copy of R with 0 adjoined in the natural way, where we write e_0 for its multiplicative unit. Then $R \cup R_0$ is naturally a layered monoid, where we define $(e_0 a)b := e_0(ab)$ and $e_0 a + e_0 b = e_0(a + b)$.

Remark 8.3. What we are lacking for obtaining a semiring is the definition of $a + b$ for $a \cong_\nu b$ with $a \in R_0$ and $b \in R_\ell$ for $\ell > 0$. The natural guess might be to define $a + b = b$ in this case, but this could ruin distributivity. If there happens to be $c \in R$ such that $bc \in R_0$, then we would have $(a + b)c = bc$, which does not necessarily equal $ac + bc$.

In this generality, we also need a more intricate definition of morphism.

Definition 8.4. A **layered morphism** of tangibly generated L -layered monoids is a map

$$\Phi := (\varphi, \rho) : (R, L, s, (\nu_{m,\ell})) \rightarrow (R', L', s', (\nu'_{m',\ell'})) \quad (8.2)$$

such that $\rho : L \rightarrow L'$ is a semiring[†] homomorphism, together with a multiplicative monoid homomorphism $\varphi : R \rightarrow R'$ that also preserves addition on $R_{>0}$ in the sense that $\varphi(a + b) = \varphi(a) + \varphi(b)$ for all a, b in $R_{>0}$, and which also satisfies the following properties:

- M1. If $\varphi(a) \notin R'_0$, then $s'(\varphi(a)) \geq \rho(s(a))$ or $s'(\varphi(a)) = 0$.
- M2. $\varphi(a^\nu) \cong_\nu \varphi(a)$.
- M3. If $a \cong_\nu b$, then $\varphi(a) \cong_\nu \varphi(b)$.

The ensuing category closely resembles the category STROP_m of [20].

8.1. Weakening the structure of L and R .

Note 8.5. To generalize the notion “supertropical semiring” from the standard supertropical theory, we could weaken Axiom A2 to:

- wA2. If $a \in R_k$ and $b \in R_\ell$, then $ab \in R_m$ for some $m \geq k\ell$.

Now we have to modify Axiom A3 to make it compatible; i.e., multiplication commutes with the sort transition maps. Technically, this says:

wA3. If $a \in R_k$ and $a' \in R_{k'}$, with $aa' \in R_{k''}$ and $\nu_{\ell,k}(a) \cdot \nu_{\ell',k'}(a') \in R_{\ell''}$ and $\nu_{m,\ell}(a) \cdot \nu_{m',\ell'}(a'') \in R_{m''}$, for $m \geq \ell$, $m' \geq \ell'$, and $m'' \geq mm'$, then
 $\nu_{q,\ell''}(aa') = \nu_{q,m''}(\nu_{m,\ell}(a) \cdot \nu_{m',\ell'}(a''))$ for all $q \geq \ell'', m''$.

This weakening is of arithmetic interest, since we now have a version of the theory without requiring a zero layer.

Remark 8.6. We do not need L to be a semiring[†], but merely a directed, partially pre-ordered multiplicative monoid (without addition). This material yields an intriguing parallel between the layered monoid R and the sorting set L (since any ordered monoid becomes a semiring[†] when addition is taken to be the maximum), and may provide guidance for future research.

Since L now is only assumed to be a multiplicative monoid, we need to remove references to addition in L . Thus, we need a formal “doubling function” $\ell \mapsto 2\ell$ on L , eliminate Axiom A4, and weaken Axiom B to:

wB. (weak supertropicality) If $a \in R_k$ and $b \in R_\ell$ with $a \cong_\nu b$, then
 $a + b \in R_m$ for some $m \geq k, \ell, \min\{2k, 2\ell\}$ with $a + b \cong_\nu b$.

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